

# ERGODICITY OF CERTAIN COCYCLES OVER CERTAIN INTERVAL EXCHANGES

DAVID RALSTON AND SERGE TROUBETZKOY

ABSTRACT. We show that for odd-valued piecewise-constant skew products over a certain two parameter family of interval exchanges, the skew product is ergodic for a full-measure choice of parameters.

## 1. INTRODUCTION AND BACKGROUND

$\mathbb{Z}$ -valued (or more generally  $G$ -valued where  $G$  is a locally compact group) skew products are a natural construction of infinite-measure preserving transformations using ergodic sums over a finite-measure preserving transformation. For a thorough overview of constructing skew products over irrational rotations, see [3]. The natural generalization of an irrational rotation is an *interval exchange transformation*; recent work in studying generic skew products over generic interval exchanges may be found in [1], where the authors establish ergodicity of skew products for step functions over generic interval exchanges. We present here an alternate ‘hands-on’ approach to prove generic ergodicity for one specific construction.

Let  $X = \mathbb{S}^1 \times \{0, 1, \dots, k-1\}$ , endowed with Lebesgue measure  $\mu$  (scaled so  $\mu(X) = k$ ), and assume that  $k = 1 \pmod 2$ . Let  $T$  be a map on  $X$  defined by

$$(1) \quad T(x, \ell) = ((x + \alpha) \bmod 1, (\ell + I(x)) \bmod k),$$

where  $I(x)$  is the characteristic function of an interval of length  $\beta$ , and  $\alpha$  is irrational;  $\{X, T\}$  is a  $\mathbb{Z}/k\mathbb{Z}$ -valued skew product (in fact a cyclic extension) of the irrational rotation by  $\alpha$ . Let  $f$  be an integer-valued function on  $X$ . The skew products we will consider are given by

$$T_f(x, \ell, m) = ((x + \alpha) \bmod 1, (\ell + I(x)) \bmod k, m + f(x, \ell)).$$

Denote by  $S_m(x, \ell)$  the  $\mathbb{Z}$ -coordinate of  $T_f^m(x, \ell, 0)$ :

$$S_m(x, \ell) = \sum_{i=0}^{m-1} f(T^i(x, \ell)).$$

Note that  $\{X \times \mathbb{Z}, T_f\}$  will *not* in general itself be a skew product over rotation by  $\alpha$ , as  $f(x, \ell)$  is not independent of  $\ell$ . We assume that  $f$  is of mean zero, and assume further that  $f$  is piecewise constant on finitely many intervals; let  $\text{Var}(f)$  be the sum over  $\ell$  of the (finite) variations of  $f$  restricted to each  $\mathbb{S}^1 \times \{\ell\}$ . Purely

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for convenience we furthermore assume that  $I$  and  $f$  are right-continuous; they are defined using intervals closed on the left and open on the right.

An integer  $E$  is an *essential value* of our skew product if for every  $A \subset X$  of positive measure, there is some  $i$  such that

$$\mu(A \cap T^i A \cap \{(x, \ell) : S_i(x, \ell) = E\}) > 0.$$

If  $E$  is an essential value, the skew product is ergodic if and only if the skew product given by  $f$  into  $\mathbb{Z}/(E\mathbb{Z})$  is ergodic.

We will use *Koksma's inequality*: let  $P$  be a partition of  $\mathbb{S}^1$  into  $q$  intervals of equal length, let  $f$  be real-valued, of bounded variation on  $\mathbb{S}^1$ , and suppose that  $x_1$  through  $x_n$  are chosen such that each interval of  $P$  contains exactly one  $x_m$ . Then

$$\left| \sum_{m=1}^n f(x_m) - n \int_{\mathbb{S}^1} f(x) dx \right| \leq \text{Var}(f).$$

Our interval exchanges are characterized by two choices:  $\alpha$  and  $\beta$ .

**Theorem 1.1.** *Let  $f$  take only odd values, and assume that not every value of  $f$  is a multiple of the same number. Then the set of  $\alpha, \beta$  for which the skew product is ergodic is of full measure.*

## 2. PROOF

**Lemma 2.1.** *Let  $f$  take integer values (not necessarily odd) and assume that not every value of  $f$  is a multiple of the same number. Further let  $\beta \in (0, 1)$  be fixed, and assume there is some finite, nonzero  $E \in \mathbb{Z}$  which is an essential value of the skew product  $\{X \times \mathbb{Z}, T_f\}$ . Then the set of  $\alpha$  for which the skew product is ergodic is of full measure.*

*Proof.* Suppose that  $\beta$  is fixed and not zero. We can construct a compact, connected translation surface  $M$  and a cross-section  $X$  so that the the first return map to  $X$  of the geodesic flow in the direction with slope  $1/\alpha$  is  $T$  given by (1) for the parameters  $\alpha, \beta$ .

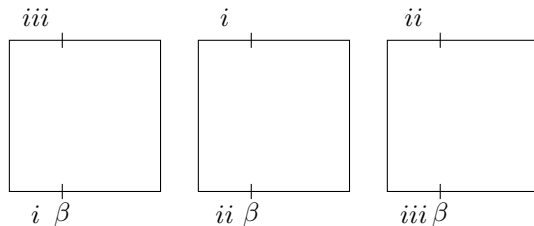


FIGURE 1. The translation surface  $M$  for  $k = 3$  and  $I(s) = 1_{[0, \beta)}$ . The unlabeled sides are identified to the opposite side in the same square, the other identifications are given by roman numbers. The cross-section  $X \times \{0, 1, 2\}$  consists of the bottom of the three squares. The flow in the vertical direction corresponds to  $\alpha = 0$ .

By [4], the system  $\{X, \mu, T\}$  is (uniquely) ergodic for almost every choice of  $\alpha$ . Now let  $X' = X \times \{0, 1, \dots, E-1\}$ , with the identification

$$(x, \ell, k) \sim (x, \ell, k + f(x, \ell) \mod E)$$

for each  $(x, \ell) \in X$ . This identification corresponds to gluing together  $E$  disjoint copies of  $M$  via the values given by  $f$ , taken modulo  $E$ ; denote this new surface by  $M'$ . So long as  $M'$  is connected, the results of [4] still apply, and the transformation

$$S'(x, \ell, k) = (x + \alpha \mod 1, \ell + I(x), k + f(x, \ell) \mod E)$$

is uniquely ergodic for almost every choice of  $\alpha$ . The assumption that the values of  $f$  generate  $\mathbb{Z}$  exactly ensure that  $M'$  is connected via Bézout's Lemma: the values taken by  $f$  on each  $X \times \{j\}$  do not depend on the choice of  $j \in \{0, 1, \dots, E-1\}$ , and there is no single common divisor for the set of values taken by  $f$ , so we may freely pass from one copy of  $M$  to another via the values of  $f$  to generate any integer value. Ergodicity of the skew product for each  $\alpha$  such that this finite system is ergodic then follows as  $E$  was assumed to be an essential value of  $\{X \times \mathbb{Z}, \mu \times dz, T_f\}$ .  $\square$

The effect of Lemma 2.1 is to reduce our problem to the existence of a single nonzero, finite essential value for generic choice of  $\beta$ . We now re-introduce the assumption that the values of  $f$  are all odd (and still not multiples of the same number). Let  $\alpha$  be irrational with continued fraction expansion

$$\alpha = [a_1, a_2, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

where each  $a_m$  is a positive integer; an excellent reference for the theory of continued fractions is [5]. Denote by  $p_n/q_n$  the convergents to  $\alpha$ , and by  $\|\cdot\|$  the distance to the nearest integer. Then it is well-known that

$$(2) \quad q_n \|q_n \alpha\| \leq \frac{1}{a_{n+1}}.$$

On  $X$  we also use  $\|\cdot\|$  for distance, with the convention that if  $\ell \neq \ell'$ ,  $\|(x, \ell) - (y, \ell')\| = 1$ . We denote by  $Q_n(T)$  the periodic approximation to  $T$  given by

$$Q_n(x, \ell) = \left( x + \frac{p_n}{q_n} \mod 1, \ell + I(x) \mod k \right).$$

**Definition 2.2.** A point  $x \in X$  will be called *n-good for rational approximation* if for all  $i = 0, 1, \dots, kq_n - 1$  we have

$$f(T^i x) = f(Q_n^i(x)), \quad I(T^i x) = I(Q_n^i x).$$

That is, as far as the functions  $f$  and  $I$  are concerned, through time  $kq_n$  we may replace the orbit of  $x$  under  $T$  with the orbit of  $x$  under  $Q_n$ .

**Definition 2.3.** A point  $x \in X$  will be called *n-spread out* if the set  $\{T^i(x)\}$ ,  $i = 0, 1, \dots, kq_n - 1$ , has the property that

- there are exactly  $q_n$  points in each  $\mathbb{S}^1 \times \{\ell\}$ , and
- for each  $\ell$ , there is a partition of  $\mathbb{S}^1 \times \{\ell\}$  into disjoint intervals of length  $1/q_n$  such that there is exactly one of the  $T^i x$  in each partition element.

**Lemma 2.4.** Suppose that  $x$  is *n-spread out*. Then

$$\left| \sum_{i=0}^{kq_n-1} f(T^i x) \right| \leq \text{Var}(f).$$

*Proof.* The restriction of the orbit of  $x$  to each  $\mathbb{S}^1 \times \{\ell\}$  may be summed separately, and the  $n$ -spread out assumption allows us to use Koksma's inequality on each  $\mathbb{S}^1 \times \{\ell\}$ .  $\square$

Let  $D = \{d_1, \dots, d_N\}$  be the projection of all discontinuities of  $f$  onto  $\mathbb{S}^1$  together with the discontinuities of  $I(x)$ . For  $n = 0 \bmod 2$  define

$$A_n = \left( \mathbb{S}^1 \setminus \left( \bigcup_{i=0}^{kq_n-1} \bigcup_{j=1}^N [d_j - k\|q_n\alpha\| - i\alpha, d_j - i\alpha] \right) \right) \times \{1, 2, \dots, k\},$$

while for  $n = 1 \bmod 2$  we use the intervals

$$[d_j - i\alpha, d_j + k\|q_n\alpha\| - i\alpha].$$

**Lemma 2.5.** *Each  $x \in A_n$  is  $n$ -good for rational approximation, and*

$$\mu(A_n) \geq k \left( 1 - k^2 N q_n \|q_n\alpha\| \right) \geq k \left( 1 - \frac{k^2 N}{a_{n+1}} \right).$$

*Proof.* The first inequality is elementary (assume all removed intervals are disjoint), and the final inequality is simply due to (2); the only content to prove is that  $x \in A_n$  implies that  $x$  is  $n$ -good for rational approximation. Suppose that  $n = 0 \bmod 2$  so that  $p_n/q_n > \alpha$ . Let  $x \in A_n$ ; there is no  $i < kq_n$  such that

$$x + i\alpha \in [d_j - k\|q_n\alpha\|, d_j].$$

The distance between  $x + i\alpha$  and  $x + ip_n/q_n$  is no larger than  $k\|q_n\alpha\|$ , so we cannot have

$$x + i\alpha < d_j \leq x + i \frac{p_n}{q_n}$$

for any  $i, j$ . As  $p_n/q_n > \alpha$ , this completes the proof for  $n = 0 \bmod 2$ . For  $n = 1 \bmod 2$  the process is identical, but we remove intervals from the other side of the discontinuities  $d_j$ , and  $p_n/q_n < \alpha$ .  $\square$

**Definition 2.6.** The action of  $T^{kq_n}$  on  $A$  is *nearly-rigid* if  $\|x - T^{kq_n}(x)\| \leq k\|q_n\alpha\|$  for all  $x \in A$ .

**Lemma 2.7.** *The action of  $T^{kq_n}$  on  $A_n$  is nearly-rigid.*

*Proof.* Through time  $q_n$  the point  $x$  orbits into the interval defining  $I(x)$  some number of times. Under  $Q_n$ , however,  $x$  has returned exactly to the same  $\mathbb{S}^1$  coordinate. Over the next  $q_n$  times, the orbit of  $x$  will therefore intersect this interval *the same number of times* (recall that  $I(x, \ell)$  is independent of  $\ell$ ), and so on for each  $q_n$  steps in the orbit. Whatever this number of intersections is, once we have applied  $Q_n$  a total of  $kq_n$  times, the total number of points in these intervals must be zero modulo  $k$ :  $Q_n^{kq_n}(x) = x$ . As  $x \in A_n$ , we certainly have  $T^{kq_n}(x)$  belonging to the same copy of  $\mathbb{S}^1$  as  $x$ , then, and the distance in  $\mathbb{S}^1$  between  $x$  and  $T^{kq_n}(x)$  is equal to  $\|kq_n\alpha\|$ , which is no larger than  $k\|q_n\alpha\|$ .  $\square$

**Definition 2.8.** The set  $A$  is *nearly invariant* under  $T$  if

$$\mu(A \triangle T(A)) \leq 2k^2 N \|q_n\alpha\|.$$

**Lemma 2.9.** *The set  $A_n$  is nearly invariant under  $T$ .*

*Proof.* Recall that  $A_n$  is constructed by removing successive preimages of  $kN$  different intervals of length  $k\|q_n\alpha\|$  ( $N$  such intervals in each copy of  $\mathbb{S}^1$ ). Therefore  $A_n \triangle T(A_n)$  at most consists of the first image of these intervals and the next preimage.  $\square$

Define

$$\sigma_n(x) = \sum_{i=0}^{q_n-1} I\left(x + \frac{i}{q_n} \bmod 1\right).$$

Note that if  $x \in A_n$ , then

$$\sigma_n(x) = \sum_{i=0}^{q_n-1} I(T^i x).$$

**Lemma 2.10.** *If  $x \in A_n$ ,  $a_{n+1} \geq k$ , and  $\sigma_n(x)$  is relatively prime to  $k$ , then  $x$  is  $n$ -spread out.*

*Proof.* Note that  $\sigma_n(x)$  is exactly the number of times through time  $q_n$  that  $I(Q_n^i x) = 1$ . By the assumption that  $x \in A_n$ , this is also the number of times that  $T^i x$  will orbit into this interval, and furthermore this number will be repeated for each successive length- $q_n$  segment of the orbit we consider:

$$x \in A_n \implies \sigma_n(x) = \sigma_n(T^{q_n} x) = \dots = \sigma_n(T^{(k-1)q_n} x).$$

As  $\sigma_n(x)$  was assumed to be relatively prime to  $k$  (i.e.  $\sigma_n(x)$  generates  $\mathbb{Z}/k\mathbb{Z}$ ), it follows that for each  $i = 0, 1, \dots, q_n - 1$ , each of

$$\{T^{i+\ell q_n}(x)\} \quad (\ell = 0, 1, \dots, k-1)$$

belongs to a *different* copy of  $\mathbb{S}^1$ . Finally, the assumption that  $a_{n+1} \geq k$  implies (again via (2)) that

$$k\|q_n\alpha\| < \frac{1}{q_n},$$

so the intervals  $[x + i/q_n, x + (i+1)/q_n)$  in each circle (if  $n = 0 \bmod 2$ ; for  $n = 1 \bmod 2$  reverse which end is closed versus open) each contain one element of the orbit.  $\square$

**Lemma 2.11.** *For all  $x$ ,  $\sigma_n(x) \in \{M, M+1\}$ , where  $M = [q_n\beta]$ , the integer part of  $q_n\beta$ .*

*Proof.* The number  $M$  is the minimum number of abutting intervals of length  $1/q_n$  (closed on the left, open on the right, say) which will always be completely contained within an interval of length  $\beta$ :

$$\frac{M}{q_n} \leq \beta < \frac{M+1}{q_n}.$$

For any  $x$ , then, there are at least  $M$  successive  $I(x + i/q_n) = 1$ . On the other hand, as  $(M+1)/q_n > \beta$ , no  $x$  may have  $\sigma_n(x) \geq M+2$ .  $\square$

**Definition 2.12.** If  $T^{kq_n}$  is nearly rigid and there is some  $\epsilon > 0$  such that  $\mu(A_n) \geq \epsilon$  then  $T$  is called *quasi-rigid* and the  $A_n$  are called *quasi-rigidity sets*.

**Corollary 2.13.** *Suppose that for infinitely many  $n$  we have*

- $a_{n+1} > k^2 N$ ,
- $q_n = 1 \bmod 2$ ,
- $\sigma_n(x)$  is relatively prime to  $k$  for all  $x \in X$ .

Then there is a finite nonzero essential value.

*Proof.* The assumption that  $a_{n+1} > k^2 N$  implies that the  $A_n$  are quasi-rigidity sets (via Lemmas 2.5 and 2.7). That  $\sigma_n(x)$  is relatively prime to  $k$  ensures that for each  $x \in A_n$ ,  $x$  is  $n$ -spread out, so by applying the Koksma inequality there is a uniform bound on the absolute value of the ergodic sums on  $A_n$ . We therefore apply [2, Corollary 2.6] (utilizing that the  $A_n$  are quasi-rigid and nearly invariant, which we have already established) to find an essential value (possibly zero) for the skew product; in short, as there is an upper bound on the sums from Koksma's inequality, we may pass to a sequence of subsets along which a single value is seen, and this value is therefore an essential value. As  $kq_n$  is odd and  $f$  takes only odd values, it follows that for all  $x \in A_n$  we must have

$$\left| \sum_{i=0}^{kq_n-1} f(T_f^i(x)) \right| \geq 1,$$

so therefore the essential value we have found in this manner is not zero.  $\square$

It is therefore of interest to determine when  $\sigma_n(x)$  is relatively prime to  $k$ .

**Lemma 2.14.** *Let  $\{m_i\}$  be an unbounded sequence of integers, and let  $k$  be a positive integer. Then for each residue class  $j \pmod k$ , for almost every  $\theta$  the equality*

$$[m_i \theta] = j \pmod k$$

*is satisfied for infinitely many  $i$ .*

*Proof.* Without loss of generality, assume that  $\{m_i\}$  are unbounded above, and by passing to a subsequence, we may assume that the  $m_i$  are *superlacunary*:

$$\lim_{i \rightarrow \infty} \frac{m_{i+1}}{m_i} = \infty.$$

Also, without loss of generality assume  $\theta \in [0, 1]$ , and define the random variable

$$X_i(\theta) = [m_i \theta] \pmod k.$$

Suppose that  $X_{i-1}(\theta) = R$ , so that for some  $M$  we have

$$\theta = \frac{R + Mk}{m_{i-1}} + \frac{\{m_{i-1}\theta\}}{m_{i-1}},$$

where  $\{x\}$  denotes the fractional part of  $x$ . The residue class of  $[m_i \theta]$ , then, is determined by the residue class of  $R'$ , where

$$\theta \in \left[ \frac{R'}{m_i}, \frac{R' + 1}{m_i} \right).$$

As the  $\{m_i\}$  are superlacunary, the number of intervals of length  $1/m_i$  within an interval of length  $1/m_{i-1}$  diverges, from which it follows that

$$\lim_{i \rightarrow \infty} \mathbb{P}(X_{i+1} = j | X_i) = \frac{1}{k}$$

for each residue class  $j$ . So along this superlacunary subsequence, for generic  $\theta$  the sequence  $[m_i \theta]$  is uniformly distributed among the residue classes, from which the lemma trivially follows.  $\square$

**Corollary 2.15.** *For almost every choice of  $\alpha, \beta$ , there are infinitely many  $n$  such that  $a_{n+1} > k^2 N$ ,  $q_n = 1 \pmod 2$ , and  $[q_n \beta] = 1 \pmod k$ .*

*Proof.* For generic  $\alpha$  there are infinitely many pairs  $a_{n+1}, a_{n+2}$  of arbitrarily large partial quotients, and no two consecutive  $q_n, q_{n+1}$  may be even, so the first two conditions are trivially satisfied. The  $\{q_n\}$  are an increasing sequence of integers, so by Lemma 2.14, for almost every  $\beta$  arbitrary residue classes of  $[t_m\beta]$  modulo any fixed  $k$  are achieved infinitely many times. □

This completes the proof of ergodicity: for generic choice of  $\alpha, \beta$  the skew product will have a nonzero essential value  $E$  by Corollary 2.13 (as  $k$  is odd, both one and two are relatively prime to  $k$ ). By Lemma 2.1, this suffices for generic ergodicity.

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BEN GURION UNIVERSITY, DEPARTMENT OF MATHEMATICS, POB 653, BEER SHEVA, 84105, ISRAEL

*E-mail address:* `ralston.david.s@gmail.com`

AIX-MARSEILLE UNIVERSITY, CPT, IML, FRUMAM, LUMINY, CASE 907, F-13288 MARSEILLE, CEDEX 09, FRANCE

*E-mail address:* `troubetz@iml.univ-mrs.fr`